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Lattice Properties of Ring-like Quantum Logics

Dietmar Dorninger,[1] Helmut Länger,[1] and Maciej Maczyński[2]

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Generalized Boolean quasirings (GBQRs) are extensions of partial algebras that are in one-to-one correspondence to bounded lattices with an involutory antiautomorphism. This correspondence generalizes the bijection between Boolean rings and Boolean algebras and provides for a large variety of presumptive quantum logics (including logics which can be defined by means of Mackey's probability function). It is shown how properties of the corresponding lattices are reflected in GBQRs and what the implications are of the associativity of the $+$-operation of GBQRs, which can be interpreted as some kind of an "exclusive or"-operation. We prove that under very weak conditions, which, however, seem to be essential for experimental verifications, the associativity of $+$ implies the classicality of the considered quantum mechanical system.

1. INTRODUCTION

Generalized Boolean quasirings (GBQRs) and partial algebras inherent to GBQRs, so-called pGBQRs, have been introduced and studied by the present authors in refs. 4–7. Similar concepts (yet somewhat more specialized) have been presented independently in refs. 2 and 8. The main purpose of our work was to provide a general framework for developing axiomatic quantum mechanics. GBQRs, which are generalizations of Boolean rings, turned out to be well suited to serve as quantum logics. Since GBQRs can be defined by means of pGBQRs, which are in one-to-one correspondence with bounded lattices with an involutory antiautomorphism, the question arises how lattice properties are reflected in pGBQRs and GBQRs and what implications properties of GBQRs have for the underlying lattices. In the

[1] Technische Universität Wien, Institut für Algebra und Computermathematik, A-1040, Vienna, Austria; e-mail: d.dorninger@tuwien.ac.at, h.laenger@tuwien.ac.at.
[2] Politechnika Warszawska, Instytut Matematyki, PL 00-661 Warsaw, Poland; e-mail: mamacz@alpha.mini.pw.edu.pl.

following we will give some answers to these questions and we will interpret our results from the quantum-logical point of view.

First, let us recall the definitions of GBQRs and pGBQRs and give a description of their connections to lattices [6, 7].

Definition 1.1. An algebra $(R, +, \cdot)$ of type (2, 2) (with two binary operations $+$ and $\cdot$) is called a *generalized Boolean quasiring* (GBQR) if there are two elements $0, 1 \in R$ such that, for all $x, y, z \in R$, the following laws hold:

(1) $x + y = y + x$
(2) $0 + x = x$
(3) $(xy)z = x(yz)$
(4) $xy = yx$
(5) $xx = x$
(6) $x0 = 0$
(7) $x1 = x$
(8) $1 + (1 + xy)(1 + x) = x$

Omitting law (1), we may consider $+$ as a partial binary operation $\oplus$ on R with domain $\{0, 1\} \times R$. This way we obtain a partial algebra $(R, \oplus, \cdot)$ of type (2, 2) (with the partial binary operation $\oplus$, and the total binary operation $\cdot$), which we will call a *partial generalized Boolean quasiring*—in short, pGBQR.

If we define in a pGBQR $(R, \oplus, \cdot)$

$$x \vee y := 1 \oplus (1 \oplus x)(1 \oplus y)$$

$$x \wedge y := xy$$

$$x^* := 1 \oplus x$$

for all $x, y \in R$, then $\mathbf{L}(R) := (R, \vee, \wedge, ^*, 0, 1)$ is a bounded lattice with an involutory antiautomorhism. Conversely, if we have a bounded lattice $(L, \vee, \wedge, ^*, 0, 1)$ with an involutory antiautomorphism * and we define

$$0 \oplus x := x$$

$$1 \oplus x := x^*$$

$$xy := x \wedge y$$

for all $x, y \in L$, then we obtain a pGBQR $\mathbf{R}(L) := (L, \oplus, \cdot)$. The above correspondences $\mathbf{L}$ and $\mathbf{R}$ establish a bijection between pGBQRs and bounded lattices with an involutory antiautomorphism. Further, every pGBQR $(R, \oplus, \cdot)$ can be extended to a GBQR by putting $0 + x = x + 0 := 0 \oplus x$ and $1 + x = x + 1 := 1 \oplus x$ for any $x \in R$ and defining $x + y = y + x \in R$ arbitrarily for all $x, y \in R \setminus \{0, 1\}$.

In the following, if not specified differently, $+$ will always refer to an arbitrary extension of $\oplus$ to a commutative total operation and $\vee$, $\wedge$, and $*$ will always mean the lattice operations on a given pGBQR or GBQR in the above-defined way. Hence, we will be looking at a pGBQR or GBQR simultanously as a ring-like structure and a lattice.

2. CHARACTERIZATION OF LATTICE PROPERTIES WITHIN pGBQRs

The following two theorems hold for arbitrary GBQRs. However, the conditions referring to GBQRs only involve the operations $+$ and $\cdot$ within the underlying pGBQRs, so that the theorems in fact are characterizations of pGBQRs.

For a GBQR $(R, +, \cdot)$, $x \leq y$ means $xy = x$, and x is said to be orthogonal to y (in symbols, $x \perp y$) if $x \leq y^*$, which is in turn equivalent to $x^* \geq y$.

Theorem 2.1. For a GBQR R and the corresponding lattice $\mathbf{L}(R)$, the following hold:

(i) $\mathbf{L}(R)$ is an ortholattice iff $x \leq y$ and $x^* \perp y^*$ together imply $x(1 + xy) = x(1 + y)$.

(ii) $\mathbf{L}(R)$ is orthomodular iff $x^* \perp y^*$ implies $x(1 + xy) = x(1 + y)$.

(iii) $\mathbf{L}(R)$ is a Boolean algebra iff $x(1 + xy) = x(1 + y)$ for all $x, y \in R$.

Proof. Let $x, y, z \in R$.

(i) "$\Rightarrow$": If $\mathbf{L}(R)$ is an ortholattice, $x \leq y$ and $x^* \perp y^*$ imply $y = 1$ because of $x, x^* \leq y$. Therefore $x(1 + xy) = 0 = x(1 + y)$.

"$\Leftarrow$": Since $x \leq 1$ and $x^* \perp 1^*$ we have $x \wedge x^* = x(1 + x1) = x(1 + 1) = 0$.

(ii) "$\Rightarrow$": If $\mathbf{L}(R)$ is orthomodular, $x^* \perp y^*$ implies $x(1 + xy) = x \wedge (x^* \vee y^*) = y^* = x(1 + y)$.

"$\Leftarrow$": According to (i), $\mathbf{L}(R)$ is an ortholattice. If $x \leq y$, then $x \perp y^*$ and hence

$$y = (x^*(1 + y))^* = (x^*(1 + x^*y))^* = x \vee (y \wedge x^*)$$

(iii) "$\Rightarrow$": If $\mathbf{L}(R)$ is a Boolean algebra,

$$x(1 + xy) = x \wedge (x^* \vee y^*) = (x \wedge x^*) \vee (x \wedge y^*) = x \wedge y^* = x(1 + y)$$

"$\Leftarrow$": According to (ii), $\mathbf{L}(R)$ is orthomodular. Now we have

$$\begin{aligned} x &= 1 + (1 + xy)(1 + x) = 1 + (1 + xy)(1 + (1 + xy)x) \\ &= 1 + (1 + xy)(1 + x(1 + xy)) = 1 + (1 + xy)(1 + x(1 + y)) \\ &= (x \wedge y) \vee (x \wedge y^*) \end{aligned}$$

which shows that any two elements of $\mathbf{L}(R)$ commute, wherefrom we can conclude [e.g., 1] that $\mathbf{L}(R)$ is a Boolean algebra. ■

Remark 2.1. If $\mathbf{L}(R)$ is a Boolean algebra, R need not be a Boolean ring, even if R has *characteristic 2* (i.e., $x + x = 0$ for all $x \in R$). See the example of the GBQR R illustrated in Fig. 1, for which we have assumed that $a + b = a$. If R were a Boolean ring, we would have to have $a + b = a + a^* = 1$.

Theorem 2.2. For a GBQR R and the corresponding lattice $\mathbf{L}(R)$, the following hold:

(i) $\mathbf{L}(R)$ is modular iff $x^* \perp y^*$ and $y \perp xz^*$ together imply $x(1 + yz) = x(1 + y)$.

(ii) $\mathbf{L}(R)$ is distributive iff $y \perp xz^*$ implies $x(1 + yz) = x(1 + y)$.

Proof. Let $x, y, z \in R$.

(i) "⇒": If $\mathbf{L}(R)$ is modular, then $x^* \perp y^*$ and $y \perp xz^*$ imply

$$x(1 + yz) = x \wedge (z^* \vee y^*) = (x \wedge z^*) \vee y^* = y^* = x(1 + y)$$

"⇐": Assume $x \leq z$. Then $z^* \perp x \vee (y \wedge z)$, and because of $x^* \wedge (y^* \vee z^*) \perp zy$ we obtain

$$\begin{aligned}(x \vee y) \wedge z &= z(1 + (x^* \wedge (y^* \vee z^*))y^*)\\ &= z(1 + (x^* \wedge (y^* \vee z^*))) = x \vee (y \wedge z)\end{aligned}$$

(ii) "⇒": In case of a distributive lattice $\mathbf{L}(R)$, $y \perp xz^*$ yields

$$x(1 + yz) = x \wedge (y^* \vee z^*) = (x \wedge y^*) \vee (x \wedge z^*) = x \wedge y^* = x(1 + y)$$

"⇐": According to (i), $\mathbf{L}(R)$ is modular. Because of $(x^* \vee z^*) \wedge y^* \perp zx$ we therefore obtain

$$\begin{aligned}(x \vee y) \wedge z &= z(1 + ((x^* \vee z^*) \wedge y^*)x^*) = z(1 + ((x^* \vee z^*) \wedge y^*))\\ &= ((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)\end{aligned}$$ ■

3. CONSEQUENCES OF THE ASSOCIATIVITY OF + IN GBQRs

Because in general there are many possibilities to extend $\oplus$ to a total operation, we start by formulating two moderate restrictions for the operation

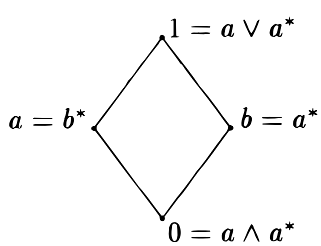

Fig. 1.

$+$ in an arbitrary GBQR $(R, +, \cdot)$ (which we will use separately as well as together).

The first restriction (A1) is that $x + y$ should be a polynomial over $(R, \oplus, \cdot)$ in the two variables x, y and the (only) constant 1 (i.e., $x + y$ should be representable by an expression in x, y, 1, $\oplus$, and $\cdot$) such that $x + y$ coincides with the symmetric difference in $\mathbf{L}(R)$ if $\mathbf{L}(R)$ is a Boolean algebra.

There are two such polynomials that represent the symmetric difference in Boolean algebras:

$$x +_1 y = 1 \oplus (1 \oplus x(1 \oplus y))(1 \oplus (1 \oplus x)y) = (x \wedge y^*) \vee (x^* \wedge y)$$

$$x +_2 y = (1 \oplus (1 \oplus x)(1 \oplus y))(1 \oplus xy) = (x \vee y) \wedge (x^* \vee y^*)$$

In general $(R, +_1, \cdot)$ and $(R, +_2, \cdot)$ are not Boolean rings unless $\mathbf{L}(R)$ is a Boolean algebra. If $\mathbf{L}(R)$ is a Boolean algebra, $+_1$ and $+_2$ coincide, otherwise $x +_1 y \leq x +_2 y$.

Our second assumption (A2) is that for an arbitrary GBQR $(R, +, \cdot)$ and $x, y \in R$

$$x +_1 y \leq x + y \leq x +_2 y$$

Assumption (A2) guarantees that if R has characteristic 2 and $\mathbf{L}(R)$ is a distributive lattice, then $\mathbf{L}(R)$ is a Boolean algebra. This can be easily seen by means of the following lemma, in which we mention several consequences of (A2) which are useful for calculations in GBQRs.

Lemma 3.1. Assume (A2). Let $a, b \in \mathbf{L}(R)$ and denote the set $\{a \wedge a^*, a, a^*, a \vee a^*\}$ by A. Then the following hold:

- (i) $a \leq b$ implies $a + b = b + a = a^* \wedge b$; hence $a + a = a \wedge a^*$ and $(a + a) + a = a$.
- (ii) $a \perp b$ implies $a + b = b + a = a \vee b$.
- (iii) $b \leq a, a^*$ implies $a + b = b + a = a$.
- (iv) $a, a^* \leq b$ implies $a + b = b + a = a^*$.
- (v) $(A, +)$ is isomorphic to the cyclic group Z_1, to the cyclic group Z_2, or to the Kleinian 4-group, depending on which of the following three conditions are fulfilled: (1) $a = a^*$, (2) $a \neq a^*$ and a, a^* are comparable, and (3) a, a^* are incomparable.

Proof. (i)–(iv) As one can see immediately, all statements hold for $+_1$ and also for $+_2$ if we replace $+$ by $+_1$ and $+_2$, respectively. Because of (A2) they are also true for $+$.

(v) One can easily check that the operation table given by Table I is valid for $+_1$ as well as $+_2$ and hence also for $+$, from which claim (v) can be derived by straightforward calculation. ∎

Table I

	$a \wedge a^*$	a	a^*	$a \vee a^*$
$a \wedge a^*$	$a \wedge a^*$	a	a^*	$a \vee a^*$
a	a	$a \wedge a^*$	$a \vee a^*$	a^*
a^*	a^*	$a \vee a^*$	$a \wedge a^*$	a
$a \vee a^*$	$a \vee a^*$	a^*	a	$a \wedge a^*$

Now we assume that in addition to (A2), $+$ is associative.

Lemma 3.2. If under the assumption (A2), $+$ is associative, then for $x, y \in R$ the following hold:

(i) $(x + y)^* = x^* + y = x + y^*$.
(ii) $x + y = x^* + y^*$.
(iii) $x + x = x + y = y + y$ implies $x = y$.
(iv) $\mathbf{L}(R)$ cannot have either of the lattices MO_2 or $MO_2 \times 2^1$ illustrated in Figs. 2 and 3 as sublattices if $a + b = z$ holds in R.

Proof. (i) $(x + y)^* = 1 + (x + y) = (1 + x) + y = x + (1 + y)$.
(ii) $x + y = (x^*)^* + y = x^* + y^*$.
(iii) Because of Lemma 3.1(i), $x + x + x = x$. Therefore

$$x = x + (x + x) = x + (y + y) = (x + y) + y = (y + y) + y = y$$

(iv) If MO_2 or $MO_2 \times 2^1$ were sublattices of $\mathbf{L}(R)$ with $a + b = z$, then we would have $a + a = a + b = b + b$, wherefrom $a = b$ follows by (iii), a contradiction. ∎

For our further investigations we need the following.

Definition 3.1. Let L be a lattice with an involutory anitautomorphism *. Then L is called **-modular* (modular in respect to the involutory antiautomorphism *) if, for all $x, y \in L$,

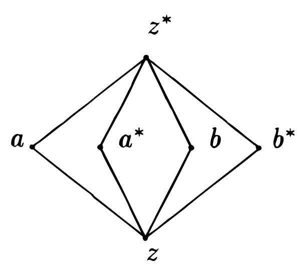

Fig. 2.

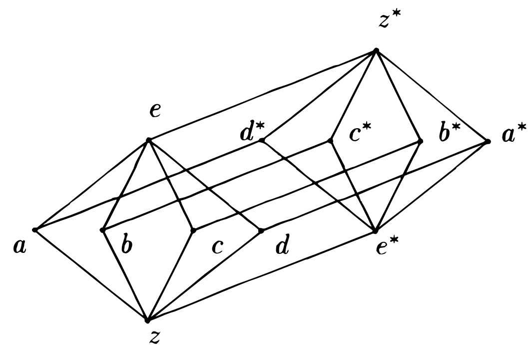

Fig. 3.

$$x \leq y \quad \text{implies} \quad (x \vee x^*) \wedge y = x \vee (x^* \wedge y) \qquad \text{(SM)}$$

If (SM) holds, then by the duality principle we also have

$$x \leq y \quad \text{implies} \quad (x \vee y^*) \wedge y = x \vee (y^* \wedge y)$$

which is then equivalent to (SM).

Moreover, we can give the following characterization:

Lemma 3.3. A lattice L with an involutory antiautomorphism * is *-modular iff for $x, y, z \in L$

$$x \leq y \leq z \quad \text{and} \quad y \leq x \vee x^* \quad \text{together imply} \quad (x \vee y^*) \wedge z = x \vee (y^* \wedge z) \qquad (\overline{\text{SM}})$$

Proof. Assume that L is *-modular. If $x \leq y \leq z$ and $y \leq x \vee x^*$, then $(y \vee y^*) \wedge z \geq (x \vee y^*) \wedge z$ and we obtain by virtue of *-modularity

$$\begin{aligned}
(x \vee y^*) \wedge z &= (((y \vee y^*) \wedge z) \vee z^*) \wedge (x \vee y^*) \wedge z \\
&= (y \vee (y^* \wedge z) \vee z^*) \wedge (x \vee y^*) \wedge z \\
&= (y \vee z^* \vee (y^* \wedge z)) \wedge (x \vee y^*) \wedge z \\
&= (((x \vee x^*) \wedge (y \vee z^*)) \vee (y^* \wedge z)) \wedge (x \vee y^*) \wedge z \\
&= (x \vee (x^* \wedge (y \vee z^*)) \vee (y^* \wedge z)) \wedge (x \vee y^*) \wedge z \\
&= ((x \vee (y^* \wedge z)) \vee (x \vee (y^* \wedge z))^*) \wedge ((x \vee y^*) \wedge z) \\
&= (x \vee (y^* \wedge z)) \vee ((x \vee (y^* \wedge z))^* \wedge ((x \vee y^*) \wedge z)) \\
&= x \vee (y^* \wedge z) \vee ((y^* \vee x) \wedge x^* \wedge z \wedge (y \vee z^*)) \\
&= x \vee (y^* \wedge z) \vee ((y^* \vee (x \wedge x^*)) \wedge z \wedge (y \vee z^*)) \\
&= x \vee (y^* \wedge z) \vee (y^* \wedge z \wedge (y \vee z^*)) = x \vee (y^* \wedge z)
\end{aligned}$$

Conversely, let us assume that ($\overline{\text{SM}}$) holds. If $x \leq y$, then $x \leq x \leq y$, which together with $x \leq x \vee x^*$ implies $(x \vee x^*) \wedge y = x \vee (x^* \wedge y)$. ∎

Remark 3.1. If a lattice with an involutory antiautomorphism * is *-modular, it must not contain a sublattice isomorphic to the lattice illustrated in Fig. 4, because from $a \geq b$ we could infer $a = a \wedge (a^* \vee b) = (a \wedge a^*) \vee b = b$.

Now we can prove that the associativity of the operation $+$ within a GBQR $(R, +, \cdot)$ has the following consequence:

Theorem 3.1. Assume (A2). If $+$ is associative, then the lattice $\mathbf{L}(R)$ corresponding to a GBQR $(R, +, \cdot)$ is *-modular. Moreover, it holds that

$$x \leq y \leq z \qquad \text{implies} \quad (x \vee y^*) \wedge z = x \vee (y^* \wedge z)$$

If R has characteristic 2, $\mathbf{L}(R)$ is orthomodular.

Proof. If $x, y, z \in R$ and $x \leq y \leq z$, then by Lemma 3.1, (i) and (ii), we obtain

$$(x \vee y^*) \wedge z = (x^* \wedge y)^* \wedge z = (x + y)^* \wedge z = (x + y) + z$$
$$= x + (y + z) = x + (y^* \wedge z) = x \vee (y^* \wedge z)$$

Hence, according to Lemma 3.3, $\mathbf{L}(R)$ is *-modular. If R has characteristic 2, $0 = x + x = x \wedge x^*$ and $1 = x \vee x^*$, which means that * is an orthocomplementation and (SM) becomes the orthomodular law. ∎

If R has characteristic 2 and $+$ is associative, Fig. 4 (Remark 3.1) shows that $\mathbf{L}(R)$ has to be orthomodular, and, according to Lemma 3.2(iv), Figs. 2 and 3 explain that $\mathbf{L}(R)$ has to be a Boolean algebra if $+ = +_1$. [Using the notation introduced in Lemma 3.2(iv), $a +_1 b = (a \wedge b^*) \vee (a^* \wedge b) = z$.] The last result has also been obtained (in greater generality) in ref. 7, Theorem 4.3.

Now we prove that under the assumptions (A1) and (A2) the last result even holds for an arbitrary operation $+$.

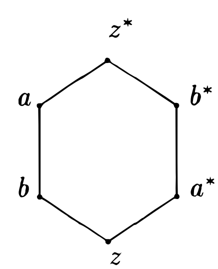

Fig. 4.

Theorem 3.2. Let $(R, +, \cdot)$ be a GBQR of characteristic 2 such that (A1) and (A2) hold. If $+$ is associative, then R is a Boolean ring and $+ = +_1 = +_2$.

Proof. According to Theorem 3.1, $\mathbf{L}(R)$ is orthomodular. As shown in ref. 3, there exist exactly six (not necessarily commutative) polynomials $x + y$ over $(R, \oplus, \cdot)$ in x, y and the (only) constant 1 such that $+$ coincides with the symmetric difference if $\mathbf{L}(R)$ is a Boolean algebra; according to ref. 3, two of these extensions are commutative, namely $+_1$ and $+_2$. Hence, because of assumption (A1), $+$ has to be either $+_1$ or $+_2$. As shown in ref. 7, associativity of $+_1$ or of $+_2$ implies $+_1 = +_2$ [7, Lemma 4.1] and that $\mathbf{L}(R)$ is a distributive lattice [7, Theorem 4.3]. Hence $\mathbf{L}(R)$ is a Boolean algebra, which corresponds to a Boolean ring. ■

Remark 3.2. It can be shown that most of the results of this paper referring to bounded lattices with an involutory antiautomorphism remain valid if one does not require the lattices to be bounded.

4. INTERPRETATION OF THE OBTAINED RESULTS WITHIN QUANTUM LOGICS

As shown in ref. 7, a quantum logic can be defined as the algebraic structure related to a system of homomorphisms induced by Mackey's probability function.

The Mackey probability function $p(A, \alpha, E)$ corresponds to the system of observables $A \in \mathbf{O}$ and states $\alpha \in S$ for a fixed physical system and is defined as the probability that a measurement of A in the state α will lead to a value in a Borel set $E \in \mathbb{B}(\mathbb{R})$. In this approach, the quantum logic corresponding to this physical system can be interpreted as a generalized Boolean quasiring $(R, +, \cdot)$, where R is the set of experimental propositions defined by

$$R := \{[(A, E)] \mid A \in \mathbf{O}, E \in \mathbb{B}(\mathbb{R})\}$$

with

$$[(A, E)] := \{(B, F) \mid B \in \mathbf{O}, F \in \mathbb{B}(\mathbb{R}), \forall \alpha \in S: p(B, \alpha, F) = p(A, \alpha, E)\}$$

In the present paper we have investigated the properties of $(R, +, \cdot)$. The operation of multiplication $\cdot$ can be defined as the unique extension of the classical "and"-operation, and its properties preserve the classical properties, as seen from axioms (3)–(7). On the other hand, the operation $p + q$ has the properties of the classical "exclusive or"-operation (p or q, but not both) only if $(R, +, \cdot)$ is a Boolean ring. In general, $+$ can be completely arbitrary in the domain $R \backslash \{0, 1\}$.

Since we would like to preserve some important properties of the classical "exclusive or"-operation also in the general case, we accept some restrictions on the arbitrariness of $+$. A first restriction is assumption (A1) stating that $x + y$ should be expressible as a polynomial in x and y over $(R, \oplus, \cdot)$. This means that the logical value of $x + y$ can be computed from x and y by a polynomial decision procedure using the fundamental operations $\oplus$ and $\cdot$ (of which $\oplus$ is a partial operation applicable only to 1 and x, and $\cdot$ is a total operation). The second assumption (A2) is that the validity of $x +_1 y$ implies the validity of $x + y$, and the validity of $x + y$ implies the validity of $x +_2 y$, where (as described above) $+_1$ and $+_2$ correspond to two possible expressions of classical "exclusive or" connected by the fundamental logical connectives "or," "and," and "not." We can say that (A2) means that the general "exclusive or" $+$ interpolates between two classical possibilities for "exclusive or," namely $+_1$ and $+_2$. This means that if $x +_1 y$ is valid and $x +_2 y$ is valid, then $x + y$ must be also valid (otherwise we would obtain a contradiction with the logical meaning of "exclusive or"). We see that (A2) is even more fundamental than (A1), since (A1) has a formal proof-theoretic character (a proof involving $+$ can be replaced by a proof involving $\oplus$ and $\cdot$ only).

If we accept (A2), or if we accept both (A1) and (A2), we can interpret our main Theorems 3.1 and 3.2 in terms of quantum logics as follows:

Interpretation of Theorem 3.1. Assume that in the ring-like quantum logic $(R, +, \cdot)$ the operation $+$ (interpreted logically as "exclusive or") interpolates between $+_1$ and $+_2$ and is associative. Then the lattice of propositions corresponding to $(R, +, \cdot)$ is *-modular, where * is the operation defined by $a^* = 1 + a$. If in addition the axiom $x + x = 0$ holds, then * is an orthocomplementation and $(R, +, \cdot)$ is orthomodular. We can say that orthomodularity follows from associativity of the "exclusive or"-operation. However, note that there are nonassociative ring-like quantum logics which are orthomodular (e.g., the Hilbert space logic).

Interpretation of Theorem 3.2. Assume that in the ring-like quantum logic $(R, +, \cdot)$ the operation $+$ is expressible by the fundamental operations $\oplus$ and $\cdot$, and $+$ interpolates between $+_1$ and $+_2$, and is associative. If in addition the axiom $x + x = 0$ holds in $(R, +, \cdot)$, then $(R, +, \cdot)$ is a Boolean ring, i.e., the logic of propositions corresponding to $(R, +, \cdot)$ is classical. Hence Theorem 3.2 can be interpreted as a characterization of classical logic in the framework of general ring-like quantum logics. In particular, if our ring-like quantum logic is nonassociative, then it is not classical. This means that if in the system of propositions corresponding to a physical system there are three propositions a, b, c such that $(a + b) + c \neq a + (b + c)$, then the logic is nonclassical. To see that $(a + b) + c \neq a + (b + c)$, it is sufficient to indicate a state $\alpha \in S$ such that for the probability p_α of the respective propositions in state α we obtain

$$p_\alpha((a + b) + c) \neq p_\alpha(a + (b + c))$$

where, for $a = [(A, E)] \in R$, $p_\alpha(a)$ is defined as $p_\alpha(a) := p(A, \alpha, E)$. If we had an apparatus to measure $a + b$ given a and b in the state $\alpha \in S$, we could easily determine $p_\alpha(a + b)$, as well as $p_\alpha((a + b) + c)$ and $p_\alpha(a + (b + c))$. If these latter probabilities are not equal, the logic (and consequently the corresponding physical system) is not classical.

Note that $a + b$ represents the exclusive "a or b," which means classically "a or b, but not both." However, the equivalence between $a + b$ and "a or b, but not both" is valid only in classical logic, so that the apparatus for measuring $a + b$ has to be constructed differently than in the classical case. It is an open experimental problem to construct such an apparatus. If we would have such an apparatus, the verification of nonclassicality of a physical system would be much simpler than observing the standard violation of the distributive law (e.g., for the momentum and position observables) which involves two logical connectives "or" and "and" instead of a single $+$ in our case. Observe that our criterion for classicality generalizes a criterion proposed in ref. 8, where the distributivity of a quasi-Boolean ring (which is a special case of our generalized Boolean quasiring, as mentioned in the introduction) is shown to be equivalent to the group structure of $(R, +)$ (hence in particular $+$ must be associative). The criterion in ref. 8 follows directly from our Theorem 3.2.

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